Exponential Functions

Finite Math

12 February 2019

Definition

Definition (Exponential Function)

An exponential function is a function of the form

$$f(x) = b^x, \ b > 0, \ b \neq 1.$$

b is called the base.

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  an imaginary number! This kind of thing will always happen if b is negative.
- If $b = 0$, then for negative x values, f is not defined. For example,

$$f(-1) = 0^{-1} = \frac{1}{0} = \text{undefined.}$$

Graphing Exponential Functions

Example

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Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x$.

Negative Powers

Notice that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$$

so that when $b < 1$, we can set $b = \frac{1}{c}$ and have $c > 1$ and

$$f(x) = b^x = \left(\frac{1}{c}\right)^x = c^{-x}.$$

So, we can always keep the base larger than 1 by using a minus sign in the exponent if necessary.

Properties of Exponential Functions

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4. b^x *is increasing if* $b > 1$.
5. b^x *is decreasing if* $0 < b < 1$.

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2. $a^x = a^y$ *if and only if* $x = y$
3. $a^x = b^x$ *for all* x *if and only if* $a = b$

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This number often shows up in growth and decay models, such as population growth, radioactive decay, and continuously compounded interest. If c is the initial amount of the measured quantity, and r is the growth/decay rate of the quantity ($r > 0$ is for growth, $r < 0$ is for decay), then the amount after time t is given by

$$A = ce^{rt}.$$

Growth and Decay Example

Example

In 2013, the estimated world population was 7.1 *billion people with a relative growth rate of* 1.1%.

(a) *Write a function modeling the world population* t *years after 2013.*

(b) *What is the expected population in 2015? 2025? 2035?*

Now You Try It!

Example

The population of some countries has a relative growth rate of 3% *per year. Suppose the population of such a country in 2012 is* 6.6 *million.*

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Solution

(a) $P = 6.6e^{0.03t}$

(b) 7.90 *million;* 8.91 *million*